JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **134**, 141–150 (1988)

A Characterization Theorem for the Fourier Transform on $\mathbf{R}^n$

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Submitted by R. P. Boas

Received December 22, 1986

In this paper we develop a characterization of the Fourier transform as a continuous operator from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$ behaving like the Fourier transform under rotations, dilation, and translations. This gives a new approach to a result of A. Hertle [6] based on the Hecke–Bochner identities. © 1988 Academic Press, Inc.

1. INTRODUCTION

Many operators in harmonic analysis exhibit simple behavior under subgroups of the group of automorphisms of $\mathbf{R}^n$. We concern ourselves here with the Fourier transform

$$f \in L^1(\mathbf{R}^n), \qquad \tilde{f}(\xi) = \int_{\mathbf{R}^n} e^{-i(x,\xi)} f(x)\, dx$$

and achieve a characterization theorem among the class of continuous operators from $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ via behavior under rotations, dilations, and translations. Denoting by $\mathcal{O}(n)$ the group of rotations in $\mathbf{R}^n$ and $f_h(x) = f(x+h)$, then the Fourier transform possesses the following well known properties:

$$\text{for} \quad A \in \mathcal{O}(n), \qquad (f \circ A)^{\sim}(\xi) = \tilde{f}(A\xi), \tag{1.1}$$

$$\text{for} \quad \lambda > 0, \qquad (f \circ \lambda)^{\sim}(\xi) = \lambda^{-n} \tilde{f}(\xi/\lambda), \tag{1.2}$$

$$\text{for} \quad h \in \mathbf{R}^n, \qquad \tilde{f}_h(\xi) = e^{i(\xi,h)} \tilde{f}(\xi). \tag{1.3}$$

This problem is not new in harmonic analysis; in fact, it has a scattered

* Presented at the first annual meeting of the International Workshop in Analysis and its Applications, June 1986, Dubrovnik-Kaspari, Yugoslavia.

0022-247X/88 $3.00

409/134/1-10

history beginning with some work of Hardy [1] and Plancherel [2] who characterized continuous operators from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$ behaving under dilations like the Fourier transform. Kober [3] and Cooper [4] characterized the Fourier transform as a continuous operator from $L^p(\mathbf{R})$ to $L^q(\mathbf{R})$ $(1<p\leqslant 2,\ 1/p+1/q=1)$ via properties (1.1), (1.2), and (1.3) as a corollary of a result in the context of functional equations. Stein [5] used this type of problem to motivate the definition of Riesz transforms, the n-dimensional analogs of the Hilbert transform. Finally, Hertle [6] proved the following result sharpening all the above, here $\mathscr{D}(\mathbf{R}^n)$ denotes the Frechet space of infinitely differentiable functions with compact support and $\mathscr{D}'(\mathbf{R}^n)$ its dual in the weak topology.

THEOREM (Hertle). *Let $T\colon \mathscr{D}(\mathbf{R}^n)\to\mathscr{D}'(\mathbf{R}^n)$ be a continuous operator. Then T is a multiple of the Fourier transform if and only if T satisfies: for $f\in\mathscr{D}(\mathbf{R}^n)$,*

(RP) *for $A\in\mathcal{O}(n)$, $[T(f\circ A)](\xi)=[Tf](A\xi)$,*

(DP) *for $\lambda>0$, $[T(f\circ\lambda)](\xi)=\lambda^{-n}[Tf](\xi/\lambda)$, and*

(TP) *for $h\in\mathbf{R}^n$, $[T(f_h)](\xi)=e^{i(\xi,h)}[Tf](\xi)$.*

What is new in this paper is our approach to the problem. Namely, we begin by expressing the continuous operator via the Schwartz kernel theorem and unraveling the nature of the kernel; the key to this unraveling is (RP) leading to the spherical and associated spherical functions on $\mathbf{R}^n$. One point of view of this approach is that it gives an apparently new derivation of the Hecke–Bochner identities (e.g., [7, 8]). This connects our work with, for example, that of Vilenkin [9] on the relation between group representation and special functions and the work of Strichartz [10] in the general context of Hecke–Bochner identities. More will be said concerning this in the final section of this paper.

2. MAIN RESULTS

Let $\mathscr{H}_m$ denote the class of (surface) spherical harmonics of degree m; $\mathscr{D}_m$ is the subspace of $\mathscr{D}(\mathbf{R}^n)$ of functions with form $f_0(r)\,Y_m(\omega)$, where $Y_m\in\mathscr{H}_m$ ($x=r\omega$ is polar notation). Notice that if $f\in\mathscr{D}_m$, $f=f_0(r)\,Y_m(\omega)$, then $f_0(r)=r^m g(r)$, where g is smooth with compact support (see, e.g., [7]). J_α and Q_α denote Bessel functions of the first and second kinds, respectively, L denotes the Laplacian on $\mathbf{R}^n$. The following is the main result embedding the method outlined in Section 1, here we assume $n>1$ reserving discussion of the case $n=1$ for the final section.

THEOREM 2.1. *Let* $n \geqslant 2$ *and let* $T: \mathscr{D}(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ *be a continuous operator. Let for* $f \in \mathscr{D}(\mathbf{R}^n)$, (RP) *and* (DP) *hold, and*

(LP) $[T(Lf)](\xi) = -|\xi|^2 (Tf)(\xi)$.

Then, there is a bounded sequence $\{c_m\}_0^\infty$ *such that for* $m \geqslant 2 - n/2$ *and* $f = f_0(\xi)\, Y_m(\omega') \in \mathscr{D}_m$,

$$(Tf)(r\omega) = c_m \left[r^{((n-2)/2)} \int_0^\infty f_0(r)\, J_{(n+2m-2)/2}(rs)\, s^{n/2}\, ds \right] Y_m(\omega). \tag{2.1}$$

Remark. For $c_m = (-i)^m$, (2.1) is precisely the Hecke–Bochner identity for the Fourier transform.

Proof. The proof is developed for $n \geqslant 3$; the case $n = 2$ requires slight modification, Chebyshev polynomials instead of Gegenbauer polynomials.

Using the Schwartz kernel theorem (e.g., [11]), there is a $K \in \mathscr{D}'(\mathbf{R}^n \times \mathbf{R}^n)$ such that for $f \in \mathscr{D}(\mathbf{R}^n)$,

$$[Tf](\xi) = \langle K(\xi, x), f(x) \rangle. \tag{2.2}$$

Applying (RP) we have for $A \in \mathcal{O}(n)$,

$$\begin{aligned} [T(f \circ A)](\xi) &= \langle K(\xi, x), f(Ax) \rangle \\ &= \langle K(\xi, A^{\mathrm{t}}x), f(x) \rangle \\ &= \langle K(A\xi, x), f(x) \rangle = [Tf](A\xi). \end{aligned}$$

(A^{t} denotes the transpose of A, of course $A^{\mathrm{t}} = A^{-1}$.) Consequently, $K(\xi, A^{\mathrm{t}}x) = K(A\xi, x)$ for all $A \in \mathcal{O}(n)$, or more appropriately:

$$\text{for} \quad A \in \mathcal{O}(n), \qquad K(A\xi, Ax) = K(\xi, x). \tag{2.3}$$

This is zonal type property which will be unraveled via:

LEMMA 2.1. *Let* $K \in \mathscr{D}'(\mathbf{R}^n \times \mathbf{R}^n)$ *such that* (2.3) *holds. Then there is a regularization* $\{K_l\}_{l=1}^\infty \subset \mathscr{D}(\mathbf{R}^n \times \mathbf{R}^n)$ (*i.e.*, $K_l \to K$ *in* $\mathscr{D}'$) *such that for all* l *and all* $A \in \mathcal{O}(n)$, $K_l(A\xi, Ax) = K_l(\xi, x)$.

The proof of this is along standard lines (see [11]), the regularization being done using radial functions. Lemma 2.1 allows one to obtain a polar representation of T. Define

$$(T_l f)(\xi) = \int_{\mathbf{R}^n} K_l(\xi, x)\, f(x)\, dx,$$

then $T_l f \to Tf$ in $\mathscr{D}'$. Let $\xi = r\omega$, $x = s\omega'$, then

$$(T_l f)(r\omega) = \int_{S^{n-1}} \int_0^\infty K_l(r\omega, s\omega')\, f(s\omega')\, s^{n-1}\, ds\, d\omega'.$$

From the zonal property of K_l, it follows that for some

$$k_l \in \mathscr{D}(\mathbf{R}_\oplus \times \mathbf{R}_\oplus \times [-1, 1]),$$
$$K_l(r\omega, s\omega') = k_l(r, s, rs(\omega, \omega')).$$

Using this and the Funk–Hecke theorem [12], we obtain for $f(s\omega') = f_0(s)\, Y_m(\omega\Xi) \in \mathscr{D}_m$:

$$\begin{aligned}(T_l f)(r\omega) &= \int_{S^{n-1}} \int_0^\infty k_l(r, s, rs(\omega, \omega'))\, f_0(s)\, Y_m(\omega')\, s^{n-1}\, ds\, d\omega' \\ &= \int_0^\infty s^{n-1} f_0(s) \left[\int_{S^{n-1}} k_l(r, s, rs(\omega, \omega'))\, Y_m(\omega')\, d\omega' \right] ds \\ &= \left\{ \int_0^\infty s^{n-1} f_0(s) \left[\frac{\omega_{n-2}}{c_n^\lambda(1)} \int_{-1}^1 k_l(r, s, rst) \right.\right. \\ &\qquad \left.\left. \times\, C_m^\lambda(t)(1-t^2)^{\lambda - 1/2}\, dt \right] ds \right\} Y_m(\omega),\end{aligned}$$

where ω_{n-2} is surface measure of S^{n-2} and C_m^λ is Gegenbauer polynomial of degree n and order $\lambda = (n-2)/2$. Denote the expression in brackets by $C_{l,m}(r, s)$ and note that $C_{l,m} \in \mathscr{D}(\mathbf{R}_\oplus \times \mathbf{R}_\oplus)$. We write

$$(T_l f)(r\omega) = \left\{ \int_0^\infty s^{n-1} C_{l,m}(r, s)\, f_0(s)\, ds \right\} Y_m(\omega).$$

Letting $l \to \infty$, it follows that there is a $C_m \in \mathscr{D}'(\mathbf{R}_\oplus \times \mathbf{R}_\oplus)$ such that

$$[Tf](r\omega) = \langle s^{n-1} C_m(r, s), f_0(s) \rangle\, Y_m(\omega). \tag{2.4}$$

We now apply (DP) to find for $\lambda > 0$,

$$C_m(\lambda r, s) = C_m(r, \lambda s). \tag{2.5}$$

Differentiating with respect to λ and substituting $\lambda = 1$, we find that distributions c satisfying (2.5) also satisfy the partial differential equation

$$r \frac{\partial c}{\partial r} = s \frac{\partial c}{\partial s}.$$

The class of distributions satisfying (2.5) is characterized in the following lemma; here, the restriction of $c \in \mathscr{D}'(\mathbf{R}_{\oplus} \times \mathbf{R}_{\oplus})$ to functions whose support lies in $\mathbf{R}_+ \times \mathbf{R}_+$ is denoted c^*.

LEMMA 2.2. *Let $C \in \mathscr{D}'(\mathbf{R}_{\oplus} \times \mathbf{R}_{\oplus})$. Then C satisfies* (2.5) *if and only if there exists $g \in \mathscr{D}'(\mathbf{R}_+)$ such that $C^*(r, s) = g(rs)$.*

Proof. Sufficiency is trivial. For necessity, consider the distribution $c^*(r/s, s)$. Then,

$$\frac{\partial}{\partial s}[c^*(r/s, s)] = -\frac{r}{s^2} c_1^*(r/s, s) + c_2^*(r/s, s)$$

$$= -\frac{1}{s}[rc_1^*(r/s, s) - sc_2^*(r/s, s)] = 0.$$

Consequently [13], $c^*(r/s, s) = g(r)$ for some $g \in \mathscr{D}'(\mathbf{R}_+)$; i.e., $c^*(r, s) = g(rs)$ as desired. We now have

$$[Tf]^*(r\omega) = \langle s^{n-1} g_m(rs), f_0(s) \rangle \, Y_m(\omega),$$

where $f = f_0(s) \, Y_m(\omega') \in \mathscr{D}_m$. We now determine g_m for the restriction $[Tf]^*$ using (LP). For this, write L in polar form

$$L = \frac{d^2}{ds^2} + \frac{n-1}{s} \frac{d}{ds} + \frac{1}{s^2} L_{sp},$$

where L_{sp} is the spherical Laplacian on S^{n-1} having the property: $L_{sp} Y_m = -m(m+n-2) \, Y_m$ for any spherical harmonic Y_m. We have,

$$(Lf.)(s\omega') = \left\{ f_0''(s) + \frac{n-1}{s} f_0'(s) - \frac{m(m+n-2)}{s^2} f_0(s) \right\} Y_m(\omega')$$

and so,

$$\begin{aligned} [T(Lf)](r\omega) &= \left\langle s^{n-1} g_m(rs), f_0''(s) + \frac{n-1}{s} f_0'(s) \right. \\ &\qquad \left. - \frac{m(m+n-2)}{s^2} f_0(s) \right\rangle Y_m(\omega) \qquad (2.6) \\ &= I_m(r) \, Y_m(\omega). \end{aligned}$$

Now,

$$\begin{aligned}\langle s^{n-1}g_m(rs), f_0''(s)\rangle &= -r\langle s^{n-1}g_m'(rs), f_0'(s)\rangle \\ &\quad -(n-1)\langle s^{n-2}g_m(rs), f_0'(s)\rangle \\ &= r^2\langle s^{n-1}g_m''(rs), f_0'(s)\rangle \\ &\quad +(n-1)\,r\langle s^{n-2}g_m'(rs), f_0(s)\rangle \\ &\quad -\left\langle \frac{n-1}{s}\,s^{n-1}g_m(rs), f_0'(s)\right\rangle.\end{aligned}$$

Consequently, from (2.6),

$$\begin{aligned}I_m(r) &= r^2\langle s^{n-1}g_m''(rs), f_0(s)\rangle + r\left\langle \frac{n-1}{s}\,s^{n-1}g_m'(rs), f_0(s)\right\rangle \\ &\quad -\left\langle \frac{m(m+n-2)}{s^2}\,s^{n-1}g_m(rs), f_0(s)\right\rangle \\ &= r^{-(n-2)}\left[\langle s^{n-1}g_m''(s), f_0(s/r)\rangle + \left\langle s^{n-1}\,\frac{n-1}{s}\,g_m'(s), f_0(s/r)\right\rangle\right] \\ &\quad -\left\langle \frac{m(m+n-2)}{s^2}\,s^{n-1}g_m(s), f_0(s/r)\right\rangle.\end{aligned}$$

Applying (LP) we obtain

$$\begin{aligned}&r^{-(n-2)}\left\langle s^{n-1}\left[g_m''(s) + \frac{n-1}{s}\,g_n'(s)\right.\right. \\ &\qquad \left.\left. +\left[1 - \frac{m(m+n-2)}{s^2}\right] g_m(s)\right], f_0(s/r)\right\rangle\, Y_m(\omega) = 0.\end{aligned}$$

It follows that g_m satisfies the following partial differential equation:

$$s^2g_m''(s) + (n-1)\,sg_m'(s) + [s^2 - m(m+n-2)]\,g_m(s) = 0.$$

This equation may be solved using a familiar device: set $g_m(s) = s^{-((n-2)/2)}h_m(s)$, then h_m satisfies Bessel's equation, $s^2h_m''(s) + sh_m''(s) + [s^2 - ((m+2m-2)/2)^2]\,h_m(s) = 0$. Consequently,

$$g_m(s) = s^{-((n-2)/2)}J_{(n+2m-2)/2}(s) \tag{2.7}$$

or

$$g_m(s) = s^{-((n-2)/2)}\,Q_{(n+2m-2)/2}(s), \tag{2.8}$$

in general g_m is a linear combination of the two. We now have that (extending $[Tf]^*$),

$$(Tf)(r\omega) = c_m\left[r^{-((n-2)/2)}\int_0^\infty f_0(s)\,J_{(n+2m-2)/2}(rs)\,s^{n/2}\,ds\right]Y_m(\omega)$$

$$+\,d_m\left[r^{-((n-2)/2)}\int_0^\infty f_0(s)\,Q_{(n+2m-2)/2}(rs)\,s^{n/2}\,ds\right]Y_m(\omega),$$

for appropriate constants c_m, and d_m. However, using Lemma 2.8 below, $d_m = 0$ for all $m \geqslant 2 - n/2$.

Remarks. (1) The sequence $\{c_m\}$ must be bounded as $\sum_{m=0}^\infty c_m q_m^2$ must converge for every $\{q_n\}_0^\infty \subset \mathbf{R}_\oplus$ such that $\sum q_n^2 < \infty$.

(2) The choice of $L^2(\mathbf{R}^n)$ in Theorem 2.1 is not important, one can use $L^p(\mathbf{R}_n)$ with any $p \geqslant 2$ and arrive at the same conclusion. In the proof, the elimination of the operator defined using (2.8) is accomplished in the following way.

LEMMA 2.3. *Let $p \geqslant 2$ and $m \geqslant 2 - n/q$. Then there exists $f \in \mathscr{D}_m$ such that $(T_1 f) \notin L^p(\mathbf{R}^n)$, where*

$$(T_1 f)(r\omega) = \left[r^{-((n-2)/2)}\int_0^\infty f_0(s)\,Q_{(n+2m-2)/2}(rs)\,s^{n/2}\,ds\right]Y_m(\omega). \quad (2.9)$$

Proof. The proof is based on the well-known asymptotic property of Q_α, i.e., [14]: for $\alpha > 0$,

$$Q_\alpha(t) \approx -\frac{2^\alpha \Gamma(\alpha)}{\pi t^\alpha} \qquad (t \to 0). \quad (2.10)$$

Let $g(s)$ be a smooth function $\operatorname{supp}(g) \subset (1, 2)$, let $f_0(s) = s^m g(s)$, and set $f = f_0(s)\,Y_m(\omega')$. We need to examine the asymptotic behavior of the expression denoted $I(r)$ in brackets in (2.9) as $r \to 0$. We have

$$I(r) = r^{-((n-2)/2)}\int_1^2 f_0(s)\,s^{m+n/2}\mathcal{Q}_{(n+2m-2)/2}(rs)\,ds$$

$$\approx -r^{-(n+m-2)}\frac{2^\alpha\Gamma(\alpha)}{\pi}\int_1^2 f_0(s)\,s^{m+n/2}\,ds \qquad (r \to 0),$$

where $\alpha = (n+2m-2)/2$. Consequently $(I(r))^p\,r^{n-1} \approx r^{-n(p-1)-mp+2p-1}$ $(r \to 0)$ and if $m \geqslant 2 - n/q$, then $|I(r)|^p\,r^{n-1} \notin L^1(0, 1)$, concluding the proof.

Remark. In the case $m<2-n/q$ ($n=2$ or 3), the operator (2.9) is bounded from $\mathscr{D}_m$ to $L^p(\mathbf{R}^n)$. These represent troublesome terms in our characterization theorem (Theorem 2.1) for dimensions less than four.

We can now prove:

THEOREM 2.2. *Let $n\geqslant 2$ and let T: $L^2(\mathbf{R}^n)$, (RP), (DP), and (TP) hold. Then $Tf=c\hat{f}$ where c is a constant.*

Proof. We assume $n\geqslant 4$ to avoid the aforementioned difficulties, these can be handled in a similar fashion. We know (Theorem 2.1) that for $f(y)=f_0(s)\, Y_m(\omega')\in\mathscr{D}_m$,

$$[Tf](r\omega)=c_m\left[r^{-((n-2)/2)}\int_0^\infty f_0(s)\, J_{(n+2n-2)/2}(rs)\, s^{n/2}\, ds\right] Y_m(\omega), \quad (2.11)$$

so we need to show $c_m=(-i)^m\, c_0$ for all n. This will be accomplished using (TP) and the function $f(y)=e^{-|y|^2}$, computing Tf_h in two ways. On one hand (Weber's exponential integral [15, p. 393]),

$$[Tf](r)=c_0 2^{-n/2}e^{-r^2/4}.$$

Consequently, using (TP),

$$[Tf_h](r\omega)=c_0 2^{-n/2}e^{ir(\omega,h)}e^{-r^2/4}.$$

Assuming $|h|=1$, expand the complex exponential in a Gegenbauer expansion convergent in L^2-norm:

$$[Tf_h](r\omega)=c_0 2^{-n/2}e^{-r^2/4}\sum_{m=0}^{\infty} q(n,m)\, r^{-((n-2)/2)}J_{(n+2m-2)/2}(r)\, C_m^\lambda((\omega,h)), \quad (2.12)$$

where $\lambda=(n-2)/2$ and $q(n,m)$ are constants dependent only on n and m. Now $f_h(y)=e^{-s^2}e^{-1}e^{-2s(\omega',h)}$, the Gegenbauer expansion for it (again convergent in L^2-norm) is

$$f_h(s\omega')=e^{-s^2}e^{-1}\sum_{m=0}^{\infty} q(n,m)(2is)^{-((n-2)/2)}\, J_{(n+2m-2)/2}(2is)\, C_m^\lambda((\omega',h)). \quad (2.13)$$

We apply T to (2.13) using (2.11):

$$(Tf_h)(r\omega)=(2i)^{-((n-2)/2)}\, e^{-1}\sum_{m=0}^{\infty} c_m q(n,m)\, I_m(r)\, C_m^\lambda((\omega,h)), \quad (2.14)$$

where

$$I_m(r) = \int_0^\infty e^{-s^2} J_{(n+2m-2)/2}(2is)\, J_{(n+2n-2)/2}(rs)\, s\, ds.$$

This is a case of Weber's second exponential integral [15, p. 495]; its value is

$$I_m(r) = \frac{e}{2}\,(i)^{((n-2)/2)}\, e^{-r^2/4} J_{(n+2m-2)/2}(r).$$

Comparing (2.14) with (2.12) gives $c_m = (-i)^n\, c_0$ as desired.

3. Additional Remarks

In the one-dimensional case our results (Theorem 2.1) takes the following form.

THEOREM 3.1. *Let $T: \mathscr{D}(\mathbf{R}) \to \mathscr{D}'(\mathbf{R})$ be a continuous operator. Let for $f \in \mathscr{D}(\mathbf{R})$,*

(DP′) *for $\lambda \neq 0$, $[T(f \circ \lambda)]\,\xi = \lambda^{-1}(Tf)(\xi/\lambda)$, and*

(LP′) $[T(f'')](\xi) = -\xi^2 (Tf)(\xi)$.

Then there are constants c_0 and c_1 such that

$$(Tf)(\xi) = c_0 \int_{\mathbf{R}} f(y)\, J_{-1/2}(\xi y)\, y^{1/2}\, dy$$

$$+ c_1 \int_{\mathbf{R}} f(y)\, Q_{-1/2}(\xi y)\, y^{1/2}\, dy. \tag{3.1}$$

The conclusion (3.1) can be written

$$(Tf)(\xi) = d_0 \tilde{f}(\xi) + d_1 \tilde{f}(-\xi).$$

One can prove the n-dimensional version of Theorem 3.1, this leads to an operator involving both Bessel functions of the first and second kind. In the latter case it would be of interest to study the mapping properties of this operator and determine in closed form the kernel K (in the proof of Theorem 3.1).

In the proof of Theorem 2.1, the Schwartz kernel theorem was used in the simplest form. Due to the use of tensor products in group representation theory (e.g., [9]) it would be interesting to use the full Schwartz kernel theorem and reformulate Theorem 2.1 or Theorem 2.2 in the lingo of group representations. This seems a tractable problem and connects our work with that of Strichartz [10] in the general context of Hecke–Bochner identities.

Finally, our methods leave open developing characterization theorems in other homogeneous spaces, e.g., the Fourier transform in hyperbolic space [9 or 16]. This latter will appear in a future paper of the author.

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